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Wavelet Zoom

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Wavelet Zoom

- Lipschitz Regularity
- Wavelet Transform Modulus Maxima
- Multiscale Edge Detection
- Multifractals

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A signal $f(t)$ is *regular* if it can be locally approximated by a polynomial.
f has a *singularity* at point v if it is not differentiable at v.

- The Fourier transform analyses the *global regularity* of a function.
- The wavelet transform makes it possible to analyze the *pointwise regularity* of a function.

Taylor polynomial approximation: $p_v(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}}{k!}(t-v)^k$

$$\Longrightarrow \quad |f(t) - p_v(t)| \leq \frac{|t-v|^m}{m!} \underset{u\in[v-h,v+h]}{sup} |f^m(u)|, \quad \forall t \in [v-h, v+h]$$

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Definition 6.1 (Lipschitz)

- A function f is pointwise Lipschitz $\alpha \geq 0$ at v, if there exists $K > 0$, and a polynomial p_v of degree $m = \lfloor \alpha \rfloor$ such that

$$\forall t \in \mathbb{R}, \quad |f(t) - p_v(t)| \leq K|t - v|^{\alpha}. \tag{6.3}$$

- A function f is uniformly Lipschitz α over $[a, b]$ if it satisfies (6.3) for all $v \in [a, b]$, with a constant K that is independent of v.
- The Lipschitz regularity of f at v or over $[a, b]$ is the supremum of the α such
 that f is Lipschitz α .

◆ If f is uniformly Lipschitz $\alpha > m$ in the neighborhood of v, then f is necessarily m times continuously differentiable in this neighborhood

◆ If the Lipschitz regularity is $\alpha < 1$ at v, then f is not differentiable at v and α characterizes the singularity type

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Fourier condition

A function f is bounded and p times continuously differentiable with bounded derivatives if

$$\int_{-\infty}^{+\infty} \left|\hat{f}(\omega)\right| (1 + |\omega|^p)\, d\omega < +\infty$$

This property is extended to Lipschitz regularity:

Theorem 6.1 A function f is bounded and uniformly Lipschitz α over $\mathbb{R}$ if

$$\int_{-\infty}^{+\infty} \left|\hat{f}(\omega)\right| (1 + |\omega|^\alpha)\, d\omega < +\infty$$

- Theorem 6.1 gives a global regularity condition.
- To get conditions on the local or even pointwise regularity of a signal, it is necessary to use a transform which is localized in time.

Wavelet transform condition

f can be approximated with a polynomial p_v in the neighborhood of v:

$$f(t) = p_v(t) + \varepsilon_v(t) \text{ with } |\varepsilon_v(t)| \leq K|t - v|^\alpha$$

Assume that the wavelet has $n > \alpha$ *vanishing moments*:

$$\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0 \quad \text{for} \quad 0 \leq k < n$$

A wavelet with n vanishing moments is orthogonal to polynomials of degree $n - 1$. Since $\alpha < n$, the polynomial p_v has degree at most $n - 1$.

$$t' = \frac{(t-u)}{s} \longrightarrow Wp_v(u, s) = \int_{-\infty}^{+\infty} p_v(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt = 0\,.$$

$$f = p_v + \varepsilon_v$$

$$Wf(u, s) = W\varepsilon_v(u, s)$$

Wavelet zoom

ψ has a fast decay: for any decay exponent $m \in \mathbb{N}$ there exists C_m such that

$$\forall t \in \mathbb{R}, \qquad |\psi(t)| \leq \frac{C_m}{1+|t|^m}$$

Theorem 6.2 A wavelet ψ with a fast decay has n vanishing moments if and only if there exists θ with a fast decay such that

$$\psi(t) = (-1)^n \frac{d^n \theta(t)}{dt^n},$$

As a consequence

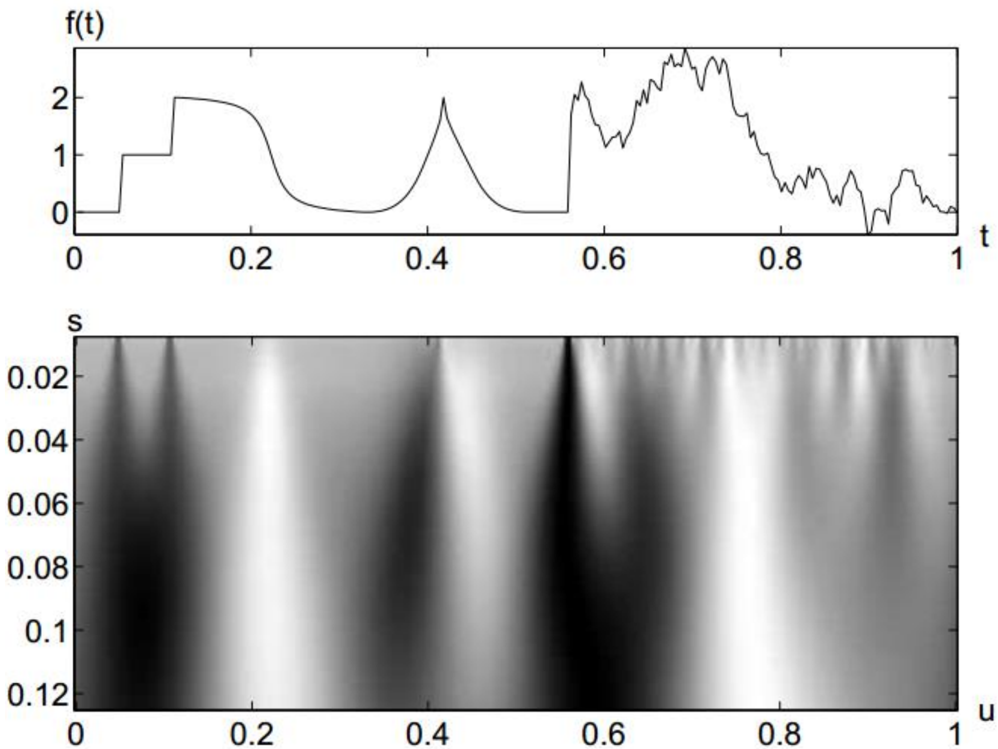
$$Wf(u,s) = s^n \frac{d^n}{du^n}(f \star \bar{\theta}_s)(u),$$

with $\bar{\theta}_s(t) = \frac{1}{\sqrt{s}}\theta\left(-\frac{t}{s}\right)$. Moreover, ψ has no more than n vanishing moments if and only if $\int_{-\infty}^{+\infty}\theta(t)dt \neq 0$.

- If $K = \int_{-\infty}^{+\infty}\theta(t)\,dt \neq 0$, then the convolution $f \star \bar{\theta}_s(t)$ can be interpreted as a weighted average of f with a kernel dilated by s.

- $Wf(u,s)$ is an nth-order derivative of an averaging of f over a domain proportional to s.

Wavelet zoom



- Wavelet transform $Wf(u,s)$ calculated with $\psi = -\theta'$ where θ is a Gaussian, for the signal f shown above.

- $Wf(u,s)$ is the derivative of f averaged in the neighborhood of u with a Gaussian kernel dilated by s.

- The position parameter u and the scale s vary respectively along the horizontal and vertical axes. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients.

- Singularities create large amplitude coefficients in their cone of influence.

Wavelet zoom

Regularity measurements with wavelets

- The decay of the wavelet transform amplitude across scales is related to the uniform and pointwise Lipschitz regularity of the signal. Measuring this asymptotic decay is equivalent to zooming into signal structures with a scale that goes to zero.

- Suppose that the wavelet ψ has n vanishing moments and is C^n with derivatives that have a fast decay. This means that for any $0 \le k \le n$ and $m \in \mathbb{N}$ there exists C_m such that

$$\forall t \in \mathbb{R}\,, \qquad \left|\psi^{(k)}(t)\right| \le \frac{C_m}{1+|t|^m}$$

Wavelet zoom

Theorem 6.3 If $f \in \mathbf{L^2}(\mathbb{R})$ is Lipschitz $\alpha \le n$ over $[a, b]$, then there exists $A > 0$ such that

$$\forall (u, s) \in [a, b] \times \mathbb{R}^+ \quad , \ |Wf(u, s)| \le A s^{\alpha + \frac{1}{2}} \tag{6.17}$$

Conversely, suppose that f is bounded and that $Wf(u, s)$ satisfies (6.17) for an $\alpha < n$ that is not an integer. Then f is uniformly Lipschitz α on $[\alpha + \varepsilon, b - \varepsilon]$, for any $\varepsilon > 0$.

- Theorem 6.3 relates the *uniform Lipschitz regularity* of f on an interval to the decay of its wavelet transform modulus at fine scales.

Wavelet zoom

Theorem 6.4 (Jaffard) If $f \in \mathbf{L}^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ at v, then there exists A such that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+ \quad , \qquad |Wf(u,s)| \leq A s^{\alpha+\frac{1}{2}} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right)$$

Conversely, if $\alpha < n$ is not an integer and there exist A and $\alpha' < \alpha$ such that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+ \quad , \qquad |Wf(u,s)| \leq A s^{\alpha+\frac{1}{2}} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right)$$

then f is Lipschitz α at v.

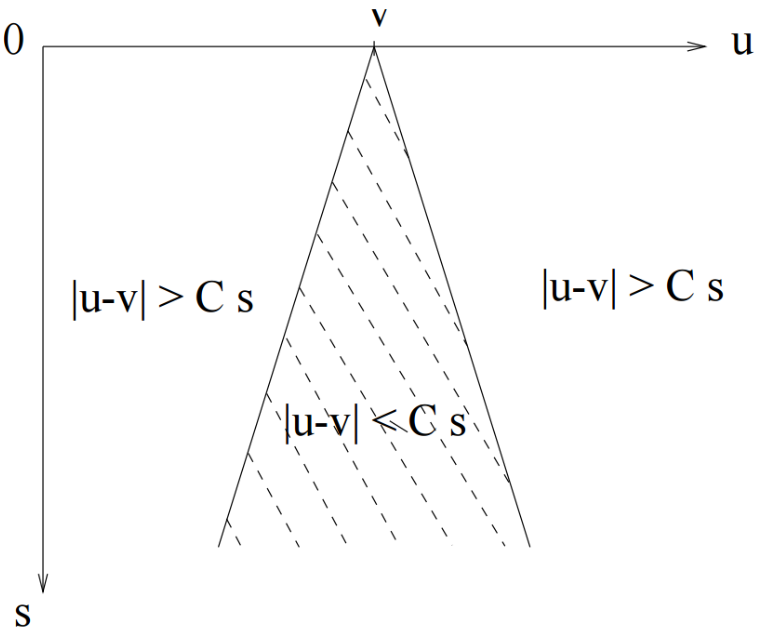
- Theorem 6.4 relates the *pointwise Lipschitz regularity* of f to the decay of its wavelet transform modulus at fine scales.
- It can be extended to an interval and to $\mathbb{R}$.

Cone of influence

Suppose that ψ has a compact support $[-C, C]$. The cone of influence of v in the scale-space plane is the set of points (u, s) such that v is included in the support of $\psi_{u,s}(t) = s^{-1/2}\psi\big((t-u)/s\big)$

Since the support of $\psi\big((t-u)/s\big)$ is $[u - Cs, u + Cs]$, the cone of influence of v is:

$$|u - v| \leq Cs.$$



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Since the support of $\psi\big((t-u)/s\big)$ is $[u - Cs, u + Cs]$, the cone of influence of v is:

$$|u - v| \leq Cs.$$

If u is in the cone of influence of v, then $Wf(u,s) = \langle f, \psi_{u,s} \rangle$ depends on the value of f in the neighborhood of v. Since $\frac{|u-v|}{s} \leq C,$ condition $|Wf(u,s)| \leq As^{\alpha+\frac{1}{2}}\left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right)$ given by theorem 6.4 can be written as:

$$|Wf(u,s)| \leq A' s^{\alpha+\frac{1}{2}}$$

which is identical to the uniform Lipschitz condition given by theorem 6.3:

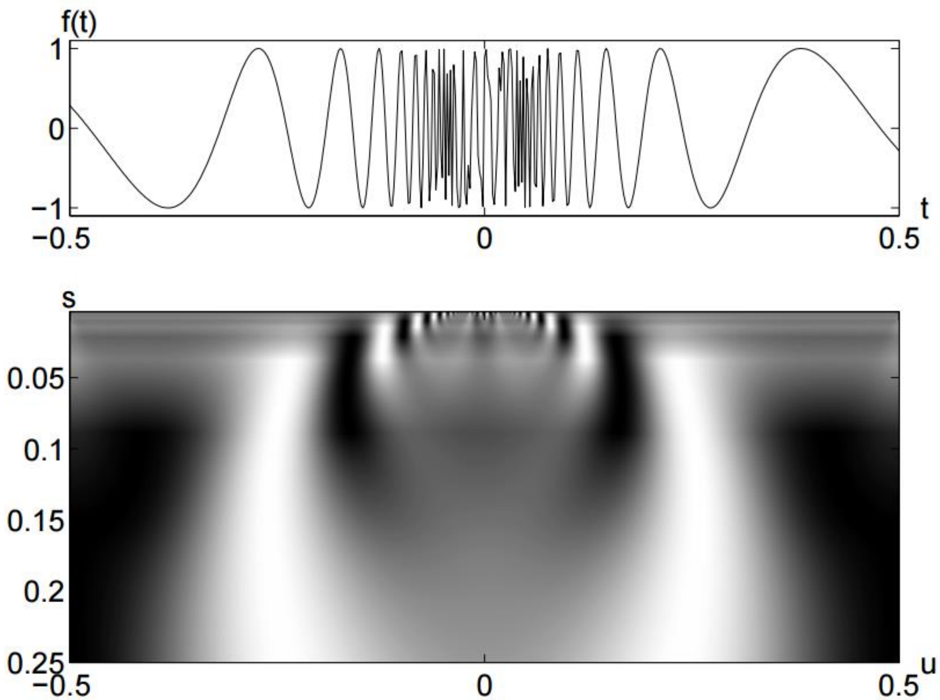
$$|Wf(u,s)| \leq As^{\alpha+\frac{1}{2}}$$

Oscillating singularities

Consider (u, s) outside of the cone of influence of v: $|u - v| > Cs$. To control the oscillations of f that might generate singularities, it is necessary to impose the decay condition when u tends to v:

$$|Wf(u,s)| \leq A' s^{\alpha - \alpha' + 1/2} |u - v|^{\alpha}$$

which guarantees that f is Lipschitz α



- Wavelet transform of $f(t) = \sin(at^{-1})$ calculated with $\psi = -\theta'$, where θ is a Gaussian.

- High-amplitude coefficients are along a parabola outside the cone of influence of $t = 0$.

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Detection of singularities

- A *wavelet modulus maximum* is defined as a point (u_0, s_0) such that $|Wf(u, s_0)|$ is locally maximum at $u = u_0$. This implies that

$$\frac{\partial Wf(u_0, s_0)}{\partial u} = 0\,.$$

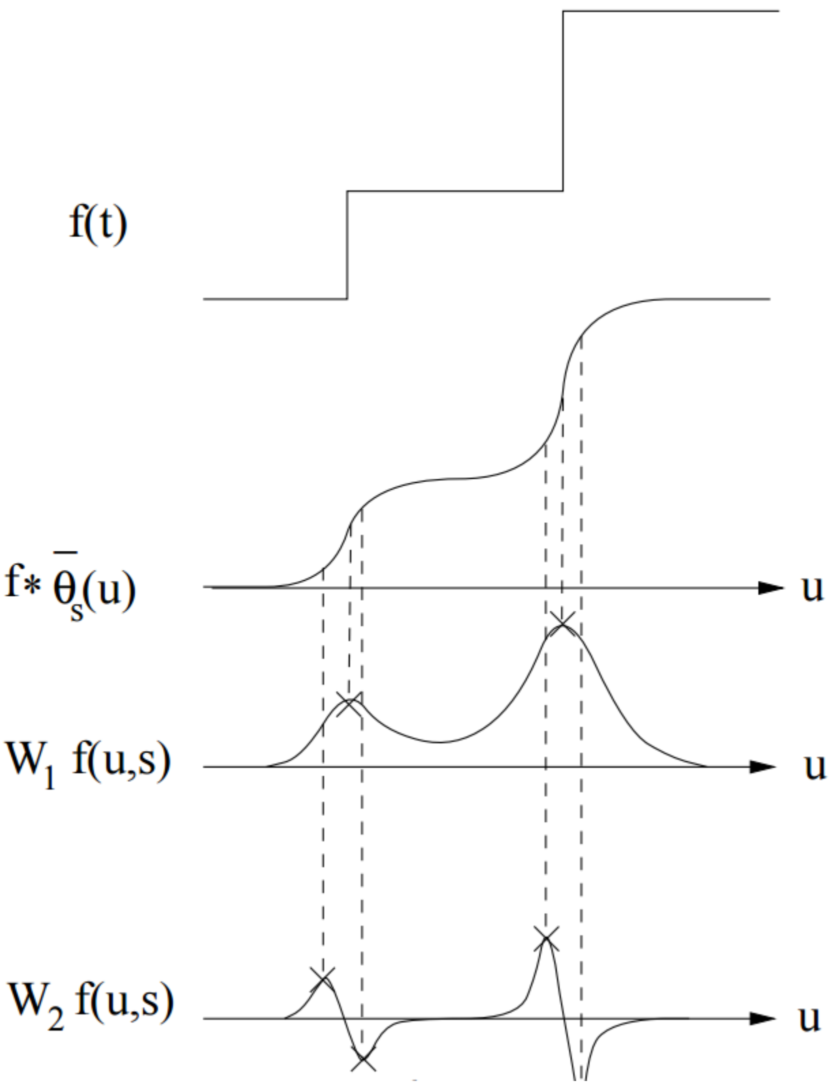
- A connected curve $s(u)$ in the scale-space plane along which all points are modulus maxima is called a *maxima line*

- Theorem 6.2 proves that if ψ has exactly n vanishing moments and a compact support, then there exists θ of compact support such that $\psi = (-1)^n \theta^{(n)}$ with $\int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0\,.$

  The wavelet transform is thus a multiscale differential operator:

$$Wf(u, s) = s^n \frac{d^n}{du^n} (f \star \bar{\theta}_s)(u).$$

Wavelet zoom



- The convolution $f \star \bar{\theta}_s(u)$ averages f over a domain proportional to s.

- If the wavelet has only one vanishing moment: $\psi = -\theta'$, then $W_1 f(u,s) = s\frac{d}{du}(f \star \bar{\theta}_s)(u)$ has modulus maxima at sharp variation points of $f \star \bar{\theta}_s\,(u)$.

- If the wavelet has two vanishing moments: $\psi = \theta''$, then the modulus maxima of $W_2 f(u,s) = s^2\frac{d^2}{du^2}(f \star \bar{\theta}_s)(u)$ correspond to locally maximum curvatures.

Wavelet zoom

Theorem 6.5 proves that if $Wf(u,s)$ has no modulus maxima at fine scales, then f is locally regular:

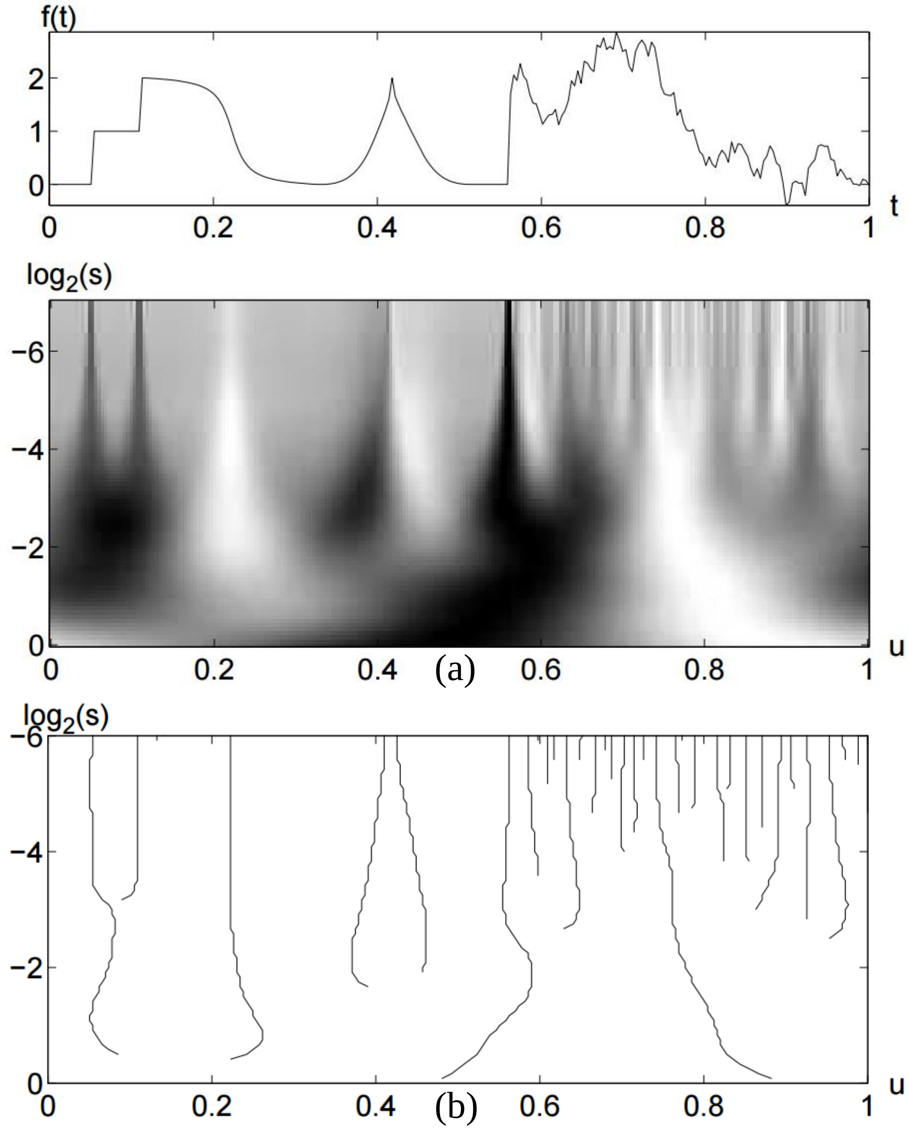
Theorem 6.5 Suppose that ψ is $\mathbf{C}^n$ with a compact support, and $\psi = (-1)^n \theta^{(n)}$ with $\int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$. Let $f \in \mathbf{L^1}[a,b]$. If there exists $s_0 > 0$ such that $|Wf(u,s)|$ has no local maximum for $u \in [a,b]$ and $s < s_0$, then f is uniformly Lipschitz n on $[\alpha + \varepsilon, b - \varepsilon]$, for any $\varepsilon > 0$.

- Theorem 6.5 implies that f can be singular (not Lipschitz 1) at a point v only if there is a sequence of wavelet maxima points $\left(u_p, s_p\right)_{p\in\mathbb{N}}$ that converges toward v at fine scales:

$$\lim_{p\to+\infty} u_p = v \quad and \quad \lim_{p\to+\infty} s_p = 0$$

- There cannot be a singularity without a local maximum of the wavelet transform at the finer scales

Wavelet zoom



(a) Wavelet transform $Wf(u,s)$. The horizontal and vertical axes give respectively u and $\log_2 s$.

(b) Modulus maxima of $Wf(u,s)$.

- All singularities are located by following the maxima lines.

Maxima propagation

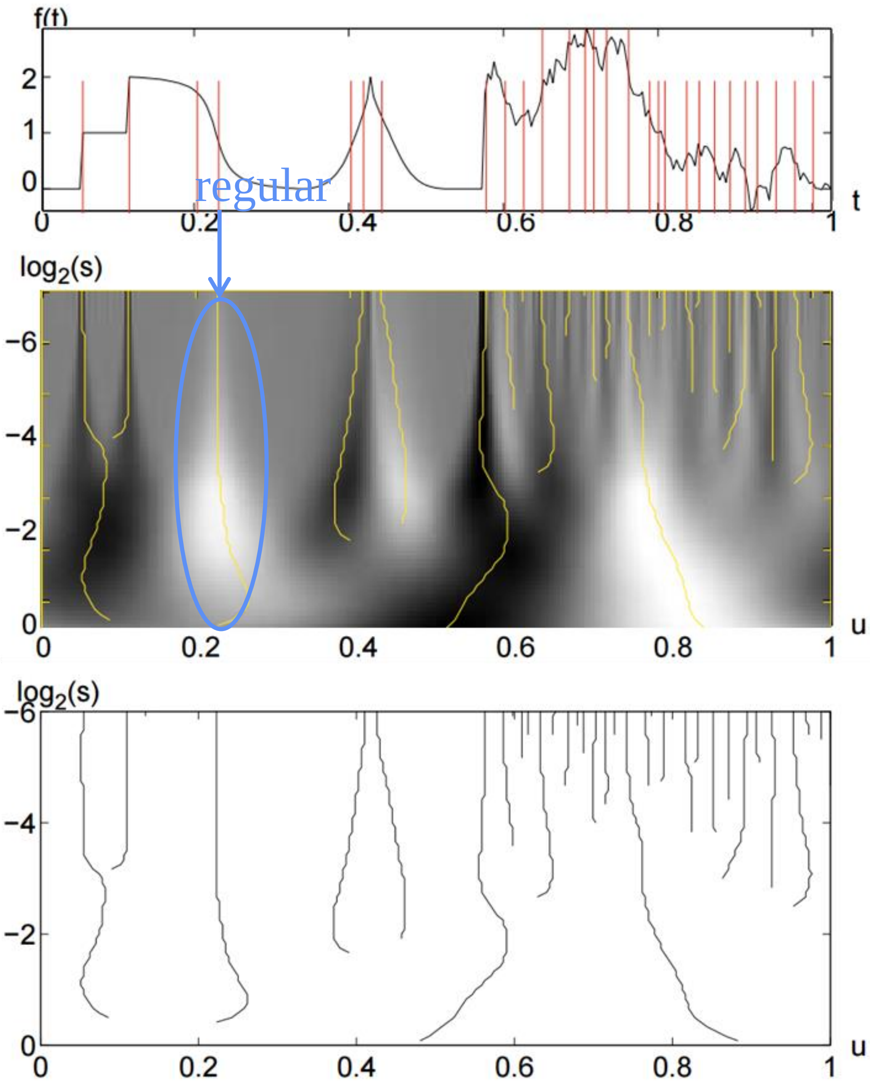
- For all $\psi = (-1)^n \theta^{(n)}$, we are not guaranteed that a modulus maxima located at (u_0, s_0) belongs to a maxima line that propagates toward finer scales. When s decreases, $Wf(u,s)$ may have no more maxima in the neighborhood of $u = u_0$

Theorem 6.6 proves that this is never the case if θ is a Gaussian:

> *Theorem 6.6* Let $\psi = (-1)^n \theta^{(n)}$, where θ is a Gaussian. For any $f \in \mathbf{L}^2(\mathbb{R})$, the modulus maxima of $Wf(u,s)$ belong to connected curves that are never interrupted when the scale decreases.

Wavelet zoom

Isolated singularities



- A wavelet transform may have a sequence of local maxima that converge to an abscissa v even though f is regular at v.

- To detect singularities it is not sufficient to follow the wavelet modulus maxima across scales

- The decay rate of the modulus maxima amplitude along the curves indicate the order of the isolated singularities (this a consequence of theorems 6.3 and 6.5)

Wavelet zoom Isolated singularities

Suppose that for $s < s_0$ all wavelet modulus maxima that converge to v are included in a cone

$$|u - v| \le Cs. \tag{6.35}$$

This means that f does not have oscillations that accelerate in the neighborhood of v. The potential singularity at v is necessarily isolated.

We can derive from Theorem 6.5 that the absence of maxima outside the cone of influence implies that f is uniformly Lipschitz n in the neighborhood of any $t \ne v$ with $t \in (v - Cs_0, v + Cs_0)$

Theorem 6.5 Suppose that ψ is $\mathbf{C}^n$ with a compact support, and $\psi = (-1)^n \theta^{(n)}$ with $\int_{-\infty}^{+\infty} \theta(t)\, dt \ne 0$. Let $f \in \mathbf{L^1}[a, b]$. If there exists $s_0 > 0$ such that $|Wf(u, s)|$ has no local maximum for $u \in [a, b]$ and $s < s_0$, then f is uniformly Lipschitz n on $[\alpha + \varepsilon, b - \varepsilon]$, for any $\varepsilon > 0$.

Wavelet zoom Isolated singularities

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Theorem 6.3 implies that f is uniformly Lipschitz α in the neighborhood of v if and only if there exists $A > 0$ such that each modulus maximum (u, s) in the cone (6.35) satisfies

$$|Wf(u,s)| \leq As^{\alpha+\frac{1}{2}}$$

Theorem 6.3 If $f \in \mathbf{L}^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ over $[a, b]$, then there exists $A > 0$ such that

$$\forall (u,s) \in [a,b] \times \mathbb{R}^+ \ \ , \ |Wf(u,s)| \leq As^{\alpha+\frac{1}{2}} \tag{6.17}$$

Conversely, suppose that f is bounded and that $Wf(u,s)$ satisfies (6.17) for an $\alpha < n$ that is not an integer. Then f is uniformly Lipschitz α on $[\alpha + \varepsilon, b - \varepsilon]$, for any $\varepsilon > 0$.

Wavelet zoom Isolated singularities

Suppose that for $s < s_0$ all wavelet modulus maxima that converge to v are included in a cone

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Theorem 6.3 implies that f is uniformly Lipschitz α in the neighborhood of v if and only if there exists $A > 0$ such that each modulus maximum (u, s) in the cone (6.35) satisfies

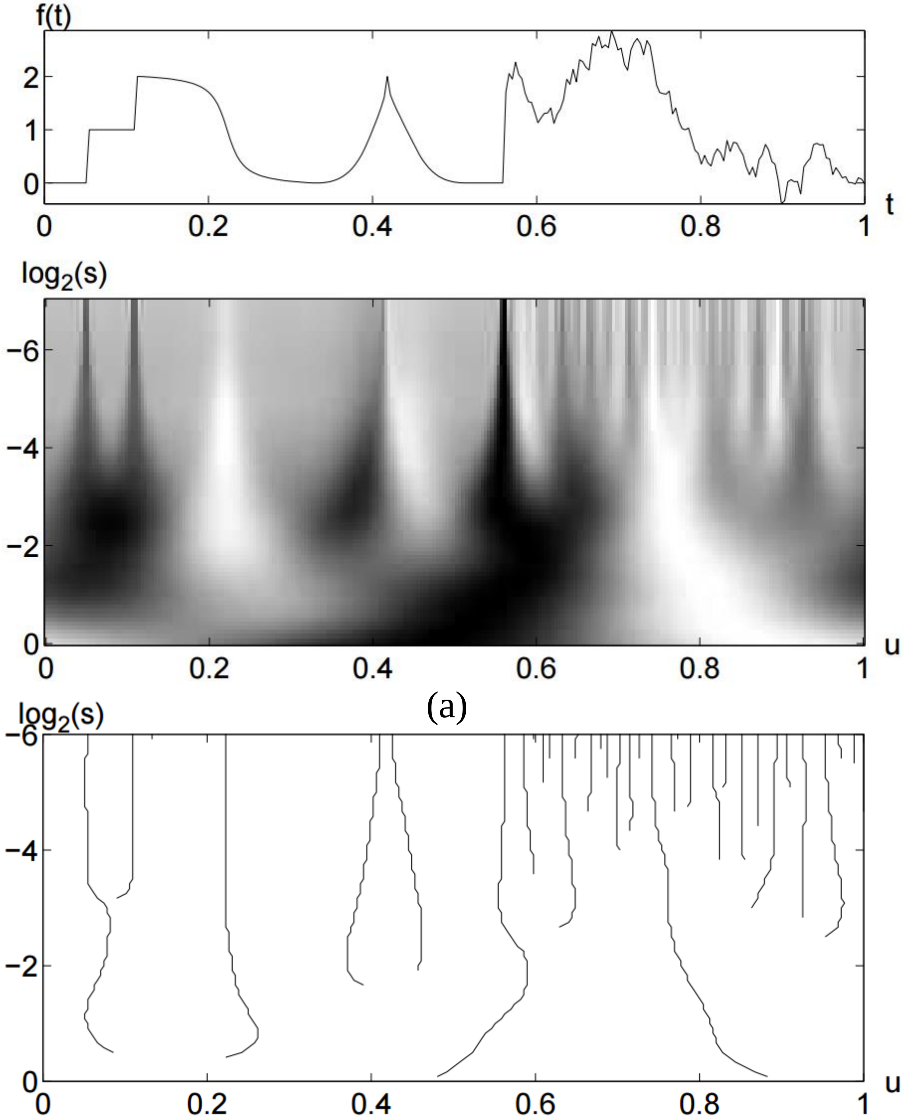
$$|Wf(u,s)| \le As^{\alpha+\frac{1}{2}}$$

which is equivalent to

$$\log_2|Wf(u,s)| \le \log_2 A + \left(\alpha + \frac{1}{2}\right)\log_2 s$$

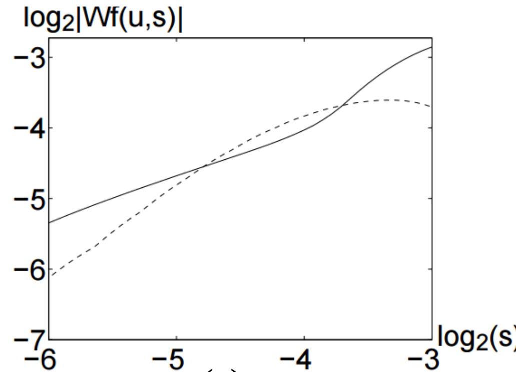
Thus, the Lipschitz regularity at v is the maximum slope of $\log_2|Wf(u,s)|$ as a function of $\log_2 s$ along the maxima lines converging to v.

Wavelet zoom Isolated singularities



(a)

(b)



(c)

(c) The full line gives the decay of $\log_2|Wf(u,s)|$ as a function of $\log_2 s$ along the maxima line that converges to the abscissa $t = 0.05$. The dashed line gives $\log_2|Wf(u,s)|$ along the left maxima line that converges to $t = 0.42$.

- For $t = 0.05$, the slope is $\alpha + 1/2 \approx 1/2$, the signal is Lipschitz 0, it has a discontinuity. For $t = 0.42$, the slope is close to $\alpha + 1/2 \approx 1$, which indicates that the signal is Lipschitz $1/2$.

Wavelet zoom

Smoothed singularities

The signal may have important variations that are infinitely continuously differentiable, e.g., at the border of a shadow the gray level of an image varies quickly but is not discontinuous because of the diffraction effect.
The smoothness of these transitions is modeled as a diffusion with a Gaussian kernel that has a variance measured from the decay of wavelet modulus maxima.
In the neighborhood of a sharp transition at v, suppose that:

$$f(t) = f_0 \star g_\sigma(t), \qquad g_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

If f_0 has a Lipschitz α singularity at v that is isolated and nonoscillating, it is uniformly Lipschitz α in the neighborhood of v. For wavelets that are derivatives of Gaussians, Theorem 6.7 relates the decay of the wavelet transform to σ and α:

Theorem 6.7 Let $\psi = (-1)^n \theta^{(n)}$ with $\theta(t) = \lambda e^{-\frac{t^2}{2\beta^2}}$. If $f = f_0 \star g_\sigma$ and f_0 uniformly Lipschitz α on $[v-h, v+h]$, then there exists A such that

$$\forall (u,s) \in [v-h, v+h] \times \mathbb{R}^+, \qquad |Wf(u,s)| \le A s^{\alpha+\frac{1}{2}} \left(1 + \frac{\sigma^2}{\beta^2 s^2}\right)^{-\frac{n-\alpha}{2}}$$

Smoothed singularities

Theorem 6.7 Let $\psi = (-1)^n \theta^{(n)}$ with $\theta(t) = \lambda e^{-\frac{t^2}{2\beta^2}}$. If $f = f_0 \star g_\sigma$ and f_0 uniformly Lipschitz α on $[v-h, v+h]$, then there exists A such that

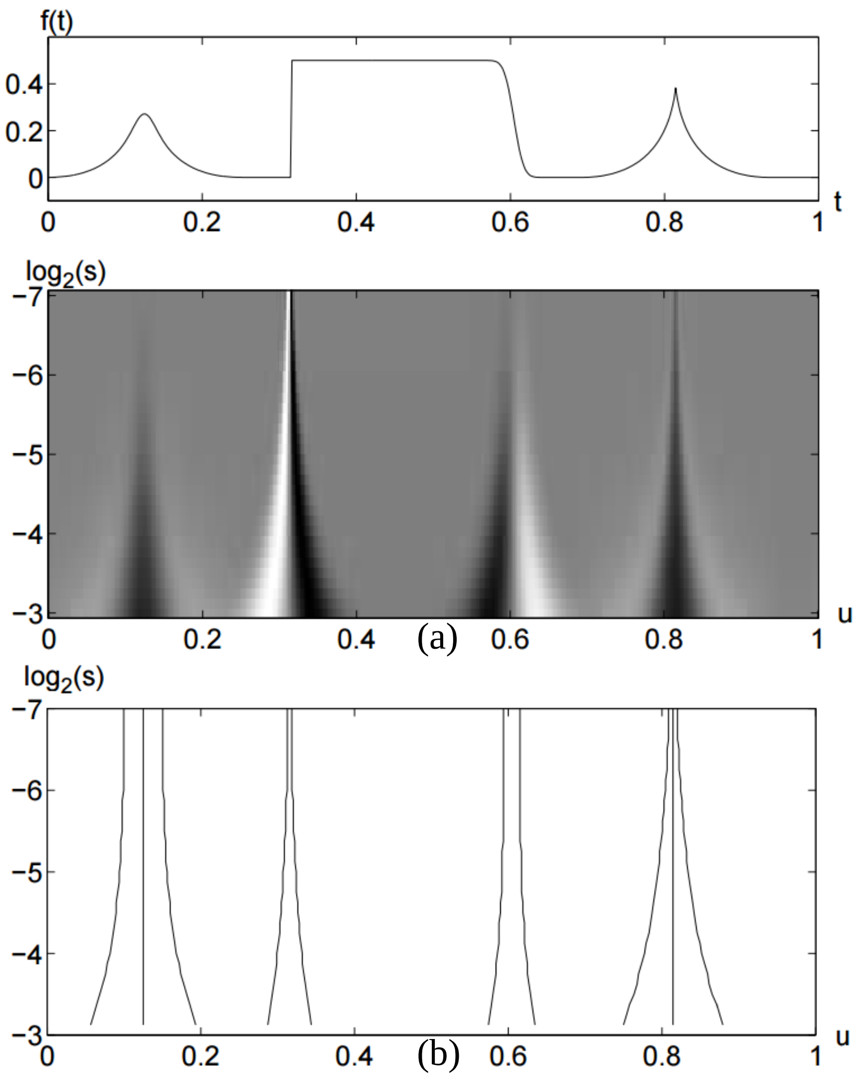
$$\forall (u,s) \in [v-h, v+h] \times \mathbb{R}^+, \qquad |Wf(u,s)| \le A s^{\alpha+\frac{1}{2}} \left(1 + \frac{\sigma^2}{\beta^2 s^2}\right)^{-\frac{n-\alpha}{2}}$$

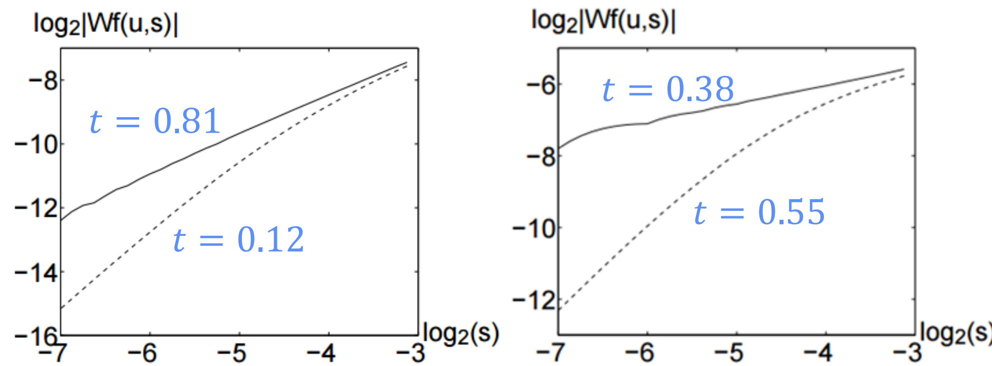
Theorem 6.7 explains how the wavelet transform decay relates to the amount of diffusion of a singularity at v:

- At large scales $s \gg \frac{\sigma}{\beta}$, the Gaussian averaging is not "felt" by the wavelet transform that decays like $s^{\alpha+\frac{1}{2}}$.
- For $s \le \frac{\sigma}{\beta}$, the variation of f at v is not sharp relative to s because of the Gaussian averaging. At these fine scales, the wavelet transform decays like $s^{n+\frac{1}{2}}$ because f is C^∞

Wavelet zoom

Smoothed singularities





(c)

(a) Wavelet transform $Wf(u,s)$.

(b) Modulus maxima of a wavelet transform computed $\psi = \theta''$, where θ is a Gaussian with variance $\beta = 1$.

(c) Decay of $\log_2|Wf(u,s)|$ along maxima curves. The diffusion at $t = 0.12$ and $t = 0.55$ modifies the decay for $s \leq \sigma = 2^{-5}$.

Dyadic maxima representation

- Wavelet transform maxima carry the properties of sharp signal transitions and singularities. By recovering a signal approximation from these maxima, signal singularities can be modified or removed by processing the wavelet modulus maxima

- For fast numerical computations, the detection of wavelet transform maxima is limited to dyadic scales $\{2^j\}_{j\in\mathbb{Z}}$. Suppose that ψ is a dyadic wavelet, which means that there exist $A > 0$ and B such that

$$\forall \omega \in \mathbb{R} - \{0\}, \qquad A \leq \sum_{j=-\infty}^{+\infty} \left|\hat{\psi}\left(2^j \omega\right)\right|^2 \leq B$$

- This dyadic wavelet transform has the same properties as a continuous wavelet transform $Wf(u,s)$. Singularities create sequence of maxima that converge toward the corresponding location at fine scales, and the Lipschitz regularity is calculated from the decay of the maxima amplitude.

Scale-space maxima support

Mallat and Zhong introduced a dyadic wavelet maxima representation with a scale-space approximation support Λ of modulus maxima $\left(u, 2^j\right)$ of Wf

Wavelet maxima can be interpreted as points of 0 or π phase for an approximate wavelet transform. Let ψ' be the derivative of ψ and $\psi'_{u,2^j}(t) = 2^{-j/2}\psi'\left(2^{-j}(t-u)\right)$. If Wf has a local extremum at u_0, then

$$\frac{\partial Wf\left(u_0, 2^j\right)}{\partial u} = -2^{-j}\left\langle f, \psi'_{2^j,u_0}\right\rangle = 0$$

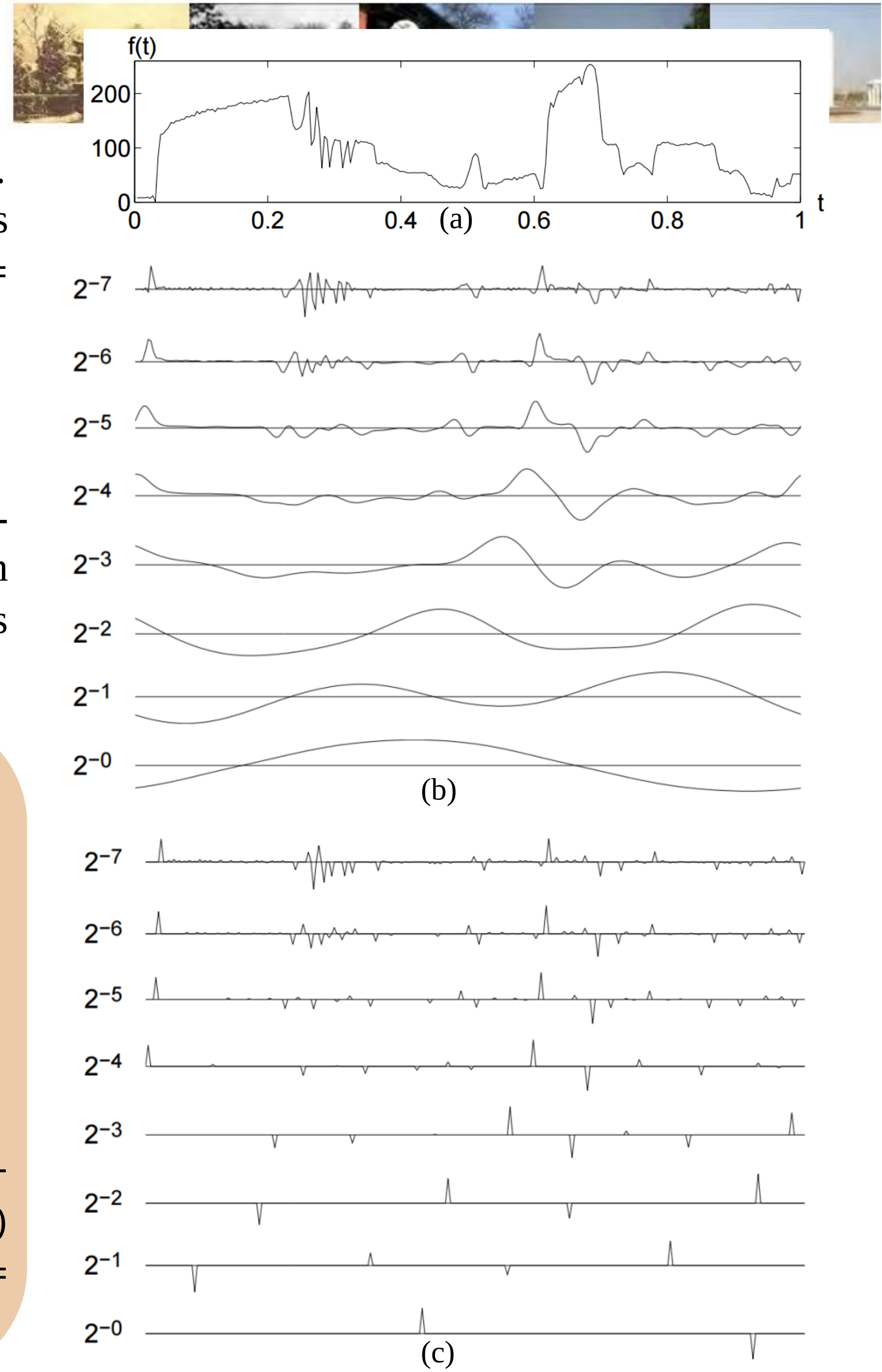
For a complex wavelet wavelet $\psi^c(t) = \psi(t) + i\psi'(t)$. If $(u,s) \in \Lambda$, then the resulting complex wavelet transform value is

$$W^c f\left(u, 2^j\right) = \left\langle f, \psi^c_{2^j,u}\right\rangle = \left\langle f, \psi_{2^j,u}\right\rangle + i\left\langle f, \psi'_{2^j,u}\right\rangle = Wf(u,s)$$

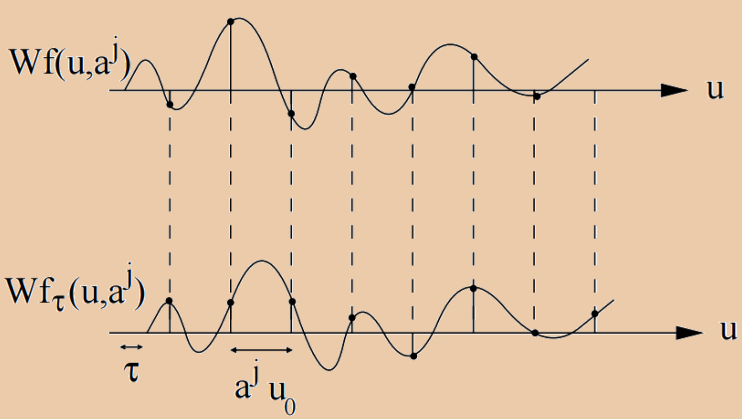
because $\left\langle f, \psi'_{2^j,u}\right\rangle = 0$. The complex wavelet value $W^c f(u,s)$ has a phase equal to 0 or π.

Wavelet zoom

(a) Intensity variation along one row of the Lena image.
(b) Dyadic wavelet transform computed at all scales $2N^{-1} \leq 2^j \leq 1$, with the quadratic spline wavelet $\psi = -\theta'$.
(c) Modulus maxima of the dyadic wavelet transform.



This adaptive sampling of u produces a translation-invariant representation. When f is translated by τ each $Wf\langle 2^j, u\rangle$ is translated by τ, so the maxima support is translated by τ:



If $f_\tau(t) = f(t-\tau)$ then $Wf_\tau(u, a^j) = Wf(u-\tau, a^j)$. Uniformly sampling $Wf_\tau(u, a^j)$ and $Wf(u, a^j)$ at $u = na^j u_0$ may yield very different values if $\tau \neq ka^j u_0$.

Approximation from wavelet maxima

- The continuous wavelet transform detects isolated singularities with their order of singularity. The regular part of the signal is coded in its coarsest approximation. We can reconstruct a signal from this coarse resolution and from its wavelet modulus maxima.

- Numerical experiments show that dyadic wavelets of compact support recover signal approximations with a relative mean square error smaller than 10^{-2}. On images, the difference is not visible.

- For general dyadic wavelets, Meyer and Berman have proved that the representation by dyadic maxima is not complete because several signals may exhibit the same wavelet maxima.

Approximation from wavelet maxima

The reconstruction of a signal from the values and scale-space locations of its wavelet modulus maxima, we can compute an orthogonal projection of f on the space generated by the complex wavelets $\left\{\psi^{c}_{u,2^j}\right\}_{(u,2^j)\in\Lambda}$.

This orthogonal projection is obtained from the dual frame $\left\{\tilde{\psi}_{u,2^j}\right\}_{(u,2^j)\in\Lambda}$ of $\left\{\psi_{u,2^j}\right\}_{(u,2^j)\in\Lambda}$ in $\mathbf{V}_\Lambda$:

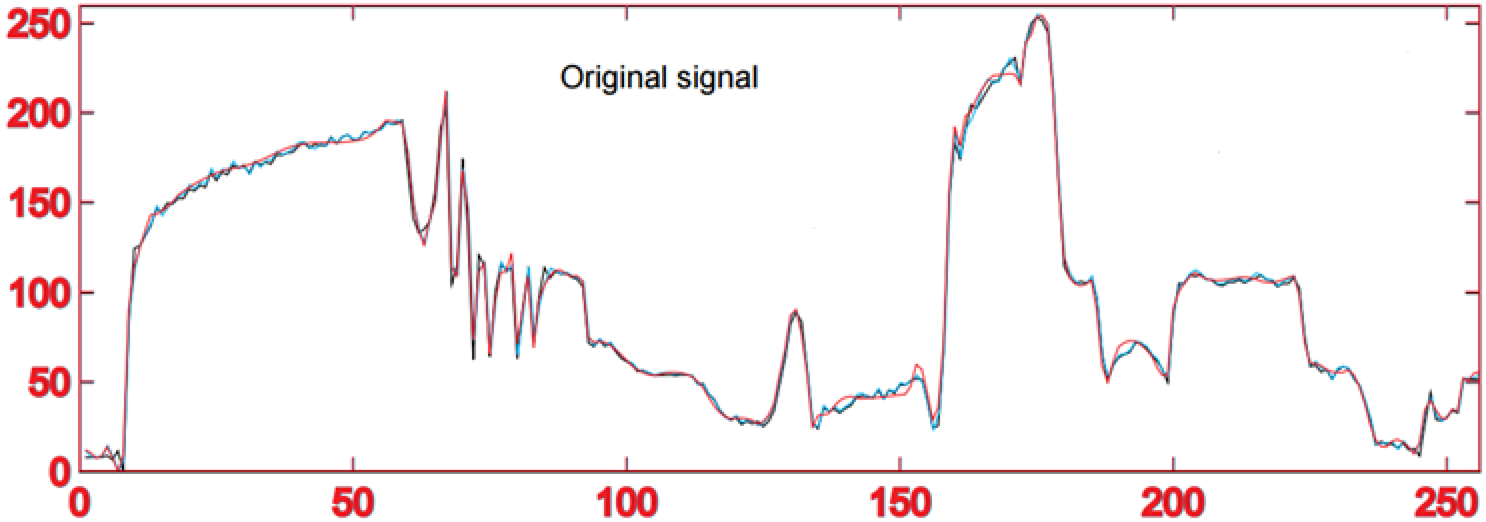
$$f_\Lambda = \sum_{(u,2^j)\in\Lambda} \left\langle f, \psi_{u,2^j}\right\rangle \tilde{\psi}_{u,2^j}$$

The dual-synthesis algorithm computes this orthogonal projection by inverting a symmetric operator L in $\mathbf{V}_\Lambda$:

$$Ly = \sum_{(u,2^j)\in\Lambda} \left\langle y, \psi_{u,2^j}\right\rangle \psi_{u,2^j} \,,$$

with a conjugate gradient algorithm. Indeed $f_\Lambda = L^{-1}(Lf)$

Approximation from wavelet maxima



- The blue line shows the approximation f_Λ of the original signal f, recovered from the dyadic wavelet maxima.

- The red line shows the approximation recovered from 50% of the wavelet maxima that have the largest amplitude. Sharp signal transitions corresponding to large wavelet maxima have not been affected, but small texture variations disappear because the corresponding maxima are removed. The resulting signal is piecewise regular

Wavelet Zoom

- Lipschitz Regularity
- Wavelet Transform Modulus Maxima
- Multiscale Edge Detection
- Multifractals

Multiscale edge detection

In images, what is most often perceived as an edge is a curve across which there is a sharp variation of brightness. To make things simpler, the image will be assumed to be monochrome. While the actual concept of an edge is more involved and depends in particular on a priori knowledge about the featured objects, this presentation has the advantage of leading to a precise mathematical definition of an "edge point".

The method of multiscale Canny edge detector is equivalent to detecting modulus maxima in a two-dimensional dyadic wavelet transform

- The scale-space support of these modulus maxima correspond to multiscale edges.
- The Lipschitz regularity of edge points is derived from the decay of wavelet modulus maxima across scales
- Image approximations are recovered with an orthogonal projection on the wavelets of the modulus maxima support

Thus, image-processing algorithms can be implemented on multiscale edges.

Canny edge detection

The canny algorithm detects points of sharp variation in an image $f(x_1, x_2)$ by calculating the modulus of its gradient vector

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

The partial derivative of f in the direction of a unit vector $\vec{n} = (\cos\alpha, \sin\alpha)$ in the $x = (x_1, x_2)$ plane is calculated as an inner product with the gradient vector

$$\frac{\partial f}{\partial \vec{n}} = \vec{\nabla} f \cdot \vec{n} = \frac{\partial f}{\partial x_1} \cos\alpha + \frac{\partial f}{\partial x_2} \sin\alpha$$

The absolute value of this derivative is maximum if $\vec{n}$ is colinear to $\vec{\nabla} f$. $\vec{\nabla} f(x)$ is parallel to the direction of maximum change of the surface $f(x)$.

A point $y \in \mathbb{R}^2$ is defined as an edge if $\left|\vec{\nabla} f(x)\right|$ is locally maximum at $x = y$ when $x = y + \lambda \vec{\nabla} f(y)$ and $|\lambda|$ is small enough. These edge points are inflection points of f.

Wavelet zoom Multiscale edge detection

A multiscale version of canny edge detector is implemented by smoothing the surface with a convolution kernel $\theta(x)$. Consider two dimensional wavelets defined by partial derivatives of θ:

$$\psi^1 = -\frac{\partial\theta}{\partial x_1} \quad \text{and} \quad \psi^2 = -\frac{\partial\theta}{\partial x_2}$$

The scale varies along the dyadic sequence $\{2^j\}_{j\in\mathbb{Z}}$. Let $x = (x_1, x_2)$, $1 \le k \le 2$

$$\psi^k_{2^j}(x_1, x_2) = \frac{1}{2^j}\psi^k\left(\frac{x_1}{2^j}, \frac{x_2}{2^j}\right) \quad \text{and} \quad \bar{\psi}^k_{2^j}(x) = \psi^k_{2^j}(-x)$$

The dyadic wavelet transform at $u = (u_1, u_2)$ is

$$W^k f\left(u, 2^j\right) = \left\langle f(x), \psi^k_{2^j}(x-u)\right\rangle = f \star \bar{\psi}^k_{2^j}(u)$$

Let $\theta_{2^j}(x) = 2^{-j}\theta\left(2^{-j}x\right)$ and $\bar{\theta}_{2^j}(x) = \theta_{2^j}(-x)$

The wavelet transform components are proportional to the gradient of f smoothed by $\bar{\theta}_{2^j}$:

$$\begin{pmatrix} W^1 f\left(u, 2^j\right) \\ W^2 f\left(u, 2^j\right) \end{pmatrix} = 2^j \begin{pmatrix} \frac{\partial}{\partial u_1}\left(f \star \bar{\theta}_{2^j}\right)(u) \\ \frac{\partial}{\partial u_2}\left(f \star \bar{\theta}_{2^j}\right)(u) \end{pmatrix} = 2^j \vec{\nabla}\left(f \star \bar{\theta}_{2^j}\right)(u)$$

Wavelet zoom Multiscale edge detection

The modulus of this gradient vector is proportional to the wavelet transform modulus:

$$Mf\left(u,2^j\right) = \sqrt{|W^1 f(u,2^j)|^2 + |W^2 f(u,2^j)|^2}$$

The angle $Af\left(u,2^j\right)$ of the wavelet transform vector:

$$Af\left(u,2^j\right) = \begin{cases} \alpha(u) & \text{if } W^1 f\left(u,2^j\right) \geq 0 \\ \pi + \alpha(u) & \text{if } W^1 f\left(u,2^j\right) \geq 0 \end{cases} \qquad a(u) = \tan^{-1}\left(\frac{W^2 f\left(u,2^j\right)}{W^1 f\left(u,2^j\right)}\right)$$

$$\overrightarrow{n_j}(u) = \left(\cos Af\left(u,2^j\right), \sin Af\left(u,2^j\right)\right)$$

An edge point v at the scale 2^j :

$Mf\left(u,2^j\right)$ is locally maximum at $u = v$ when $u = v + \lambda \vec{n}_j(v)$ and $|\lambda|$ small enough.

These points are also called wavelet transform modulus maxima.

Wavelet zoom Multiscale edge detection

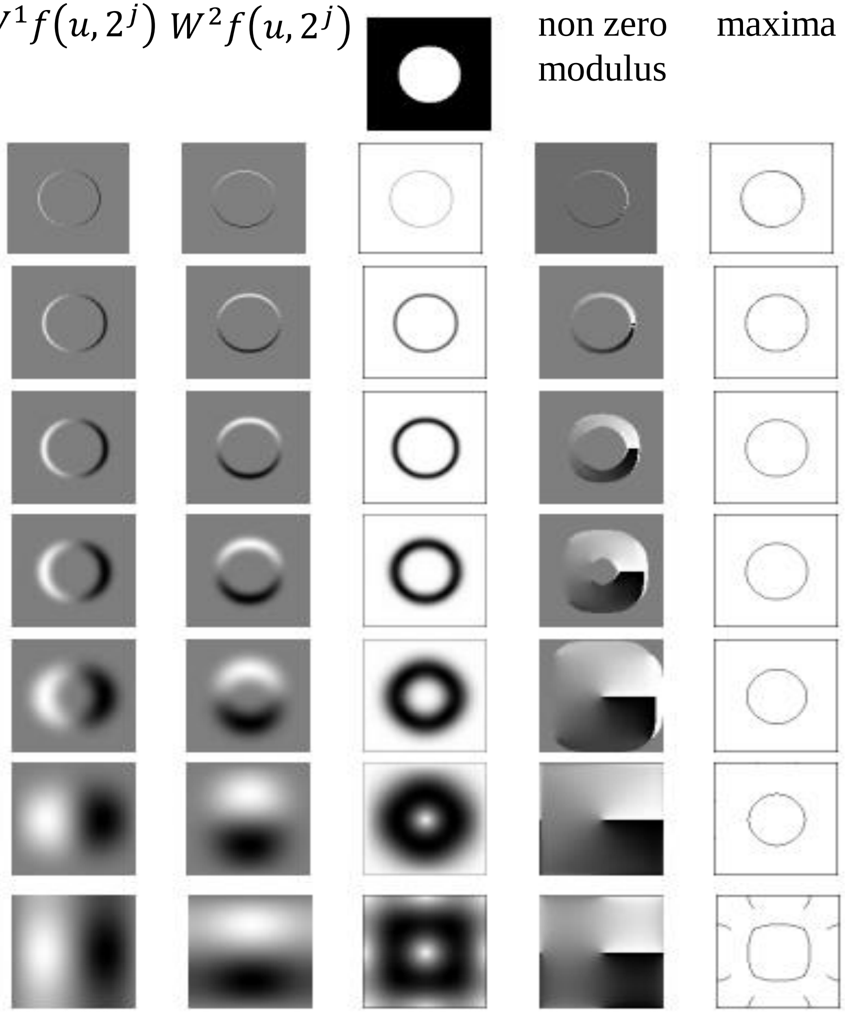
Horizontal wavelet transform $W^1 f(u, 2^j)$

Vertical wavelet transform $W^2 f(u, 2^j)$

Wavelet transform modulus

Wavelet transform angle for a non zero modulus

Wavelet transform modulus maxima



- The original image is on top
- The wavelet transform has a scale 2^j $(-6 \le j \le 0)$ that increases from top to bottom

Wavelet zoom

Maxima curves

- Edge points are distributed along curves that often correspond to the boundary of important structures. Individual wavelet modulus maxima are chained together to form a maxima curve that follows an edge

- At any location, the tangent of the edge curve is approximated by computing the tangent of a level set. This tangent direction is used to chain wavelet maxima that are along the same edge curve

The level sets of $g(x)$ are the curves $x(s)$ in the (x_1, x_2) plane where $g\big(x(s)\big)$ is constant. s is the arc-length of the level set. Let $\vec{\tau} = (\tau_1, \tau_2)$ be the direction of the tangent of $x(s)$. Since $g\big(x(s)\big)$ is constant when s varies:

$$\frac{\partial g\big(x(s)\big)}{\partial s} = \frac{\partial g}{\partial x_1}\tau_1 + \frac{\partial g}{\partial x_2}\tau_2 = \vec{\nabla} g \cdot \vec{\tau} = 0$$

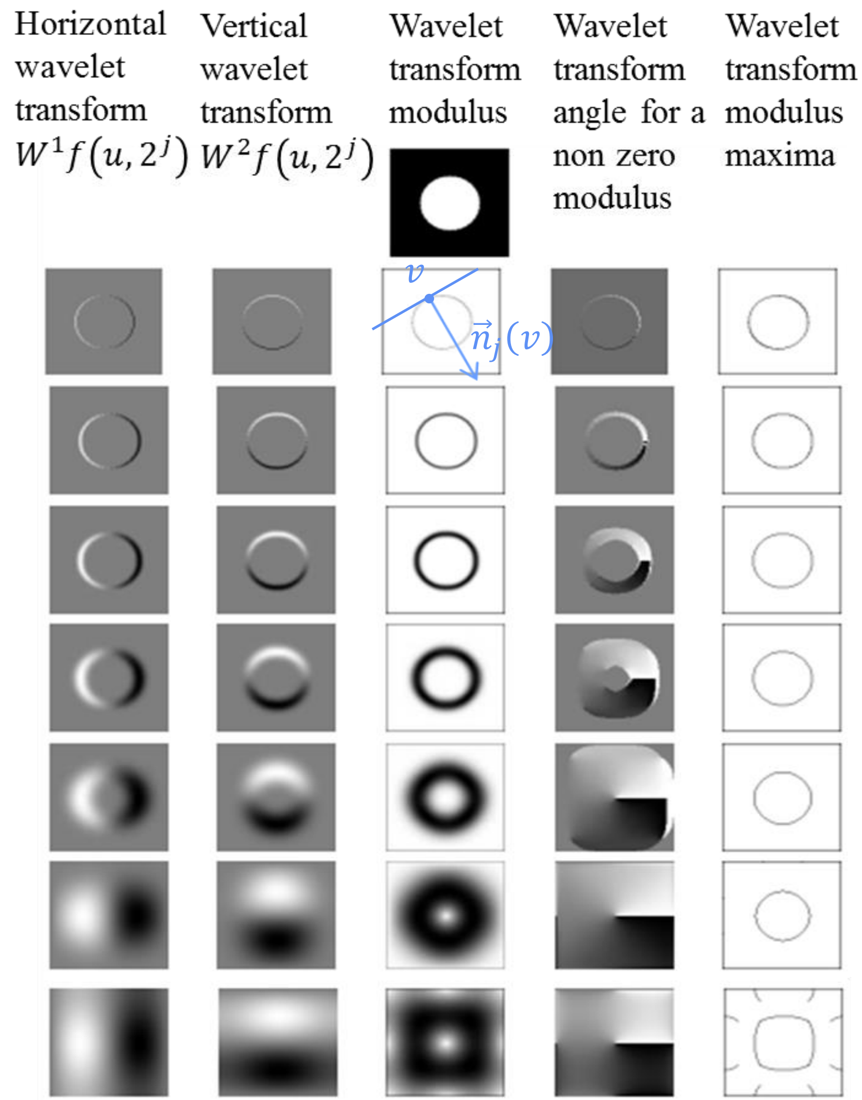
$\vec{\nabla} g(x)$ is perpendicular to the direction $\vec{\tau}$ of the tangent of the level set that goes through x.

Wavelet zoom

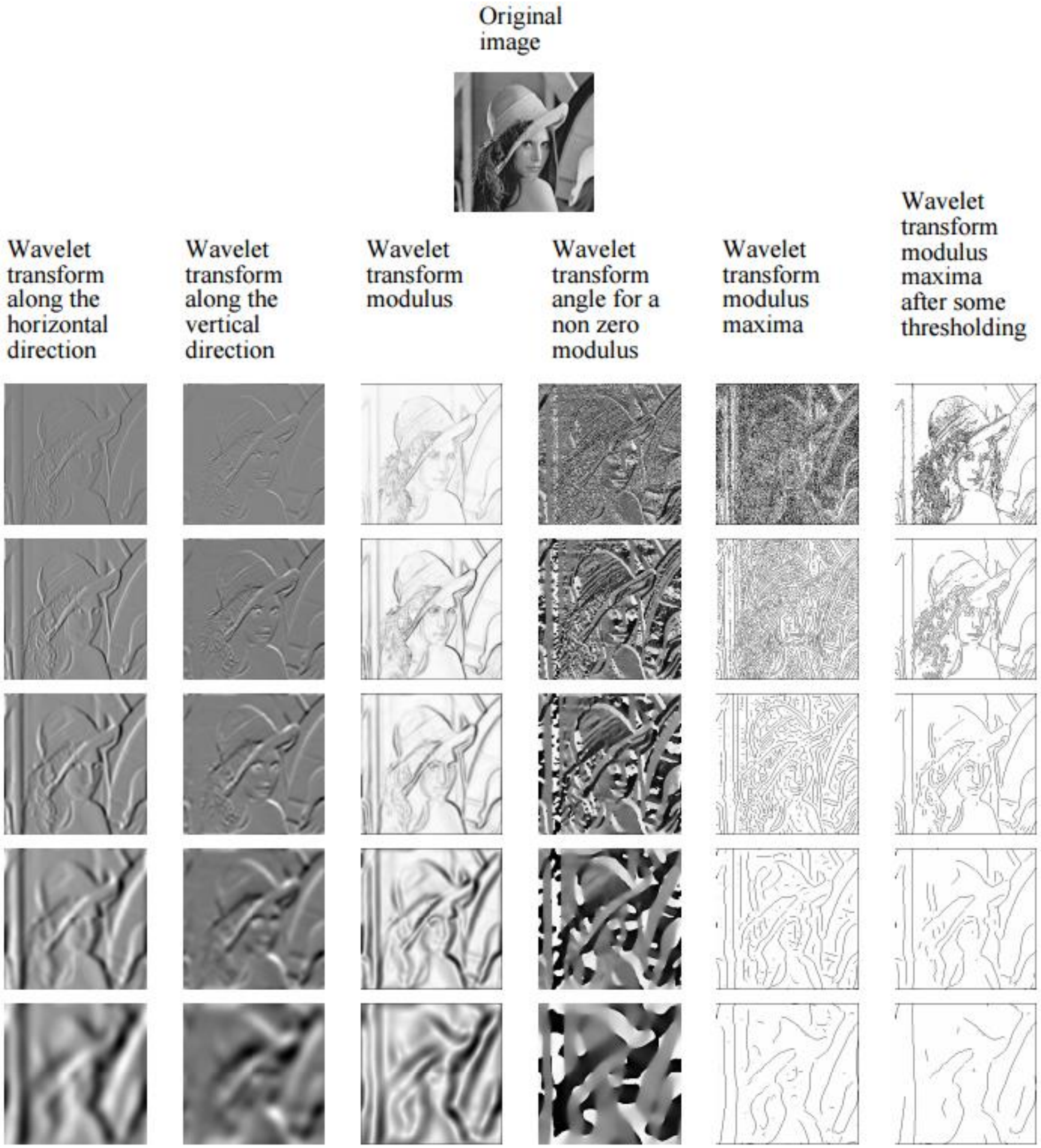
Maxima curves

- The level set property applied to $g = f \star \bar{\theta}_{2^j}$ proves that at a maximum point v the vector $\vec{n}_j(v)$ of angle $Af(v, 2^j)$ is perpendicular to the level set of $f \star \bar{\theta}_{2^j}$ going through v.

- If the intensity profile remains constant along an edge, then the inflection points (maxima points) are along a level set. The intensity profile of an edge may not be constant but its variations are often negligible over a neighborhood of size 2^j for a sufficiently small scale 2^j. The tangent of the maxima curve is then nearly perpendicular to $\vec{n}_j(v)$



Wavelet zoom



- The wavelet transform has a scale 2^j $(-7 \leq j \leq -3)$ that increases from top to bottom

- Some edges disappear when the scale increases. These correspond to fine-scale intensity variations that are removed by the average of $\bar{\theta}_{2^j}$ when 2^j is large. The averaging also modifies the position of the remaining edges.

Wavelet zoom

Reconstruction from edges

Image approximations can be computed by projecting the image on the space generated by wavelets on the modulus maxima support

Let Λ be the set of all modulus maxima points $(u, 2^j)$, $\vec{n}$ is the unit vector in the direction of $Af(u, 2^j)$ and

$$\psi^3_{u,2^j}(x) = 2^{2j} \frac{\partial^2 \theta_{2^j}(x-u)}{\partial \vec{n}^2}$$

Since the wavelet transform modulus $Mf(u, 2^j)$ has a local extremum at u in the direction of $\vec{n}$:

$$\left\langle f, \psi^3_{u,2^j} \right\rangle = 0$$

A modulus maxima representation provides the set of inner products $\left\{\left\langle f, \psi^k_{u,2^j} \right\rangle\right\}_{(u,2^j)\in\Lambda, 1\le k\le 3}$. A modulus maxima approximation f_Λ can be computed as an orthogonal projection of f on the space generated by the family of maxima wavelets $\left\{\psi^k_{u,2^j}\right\}_{(u,2^j)\in\Lambda, 1\le k\le 3}$.

Reconstruction from edges

A modulus maxima approximation f_Λ can be computed as an orthogonal projection of f on the space generated by the family of maxima wavelets $\left\{\psi^k_{u,2^j}\right\}_{(u,2^j)\in\Lambda, 1\le k\le 3}$.

The dual-synthesis algorithm computes this orthogonal projection by inverting a symmetric operator L in $\mathbf{V}_\Lambda$:

$$Ly = \sum_{(u,2^j)\in\Lambda} \sum_{k=1}^{2} \left\langle y, \psi^k_{u,2^j} \right\rangle \psi^k_{u,2^j},$$

with a conjugate gradient algorithm. Indeed $f_\Lambda = L^{-1}(Lf)$.

When keeping all modulus maxima, the resulting approximation f_Λ satisfies $\|f_\Lambda - f\| / \|f\| \le 10^{-2}$.

Singularities and edges are nearly perfectly recovered and no spurious oscillations are introduced. The image differ slightly in smooth regions, but visually this is not noticeable.

Wavelet zoom

Reconstruction from edges

Original Lena image

Reconstruction from modulus maxima and coarse approximation

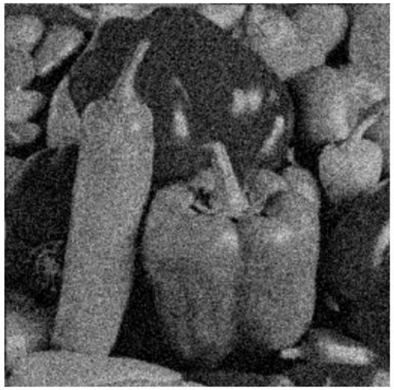
Reconstruction from thresholded modulus maxima and coarse approximation. The thresholding accounts for the disappearance of image structures. Sharp image variations are recovered

Wavelet zoom

Denoising by multiscale edge thresholding

- Multiscale edge representations can be used to reduce additive noise. Noise can be removed by thresholding multiscale wavelet maxima, while taking into account their geometric properties. The following example chains the maxima into curves that are thresholded as a block:

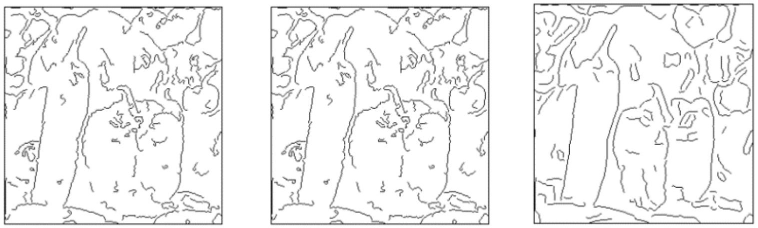
Noisy peppers image



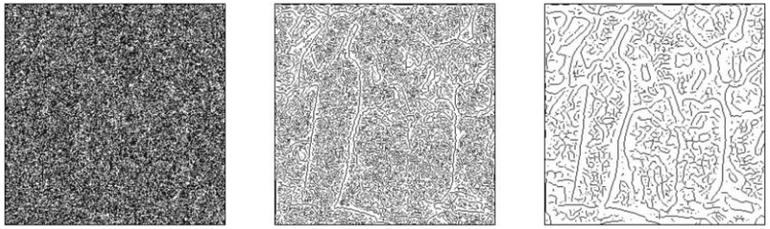
Peppers image restored from the thresholding maxima chains

Edges are well-recovered visually but textures and fine structures are removed, producing a cartoonlike image

Maxima support computed with a thresholding selection of the maxima chains



Wavelet maxima support of the noisy image, the scale increases from left to right, from 2^{-7} to 2^{-5}

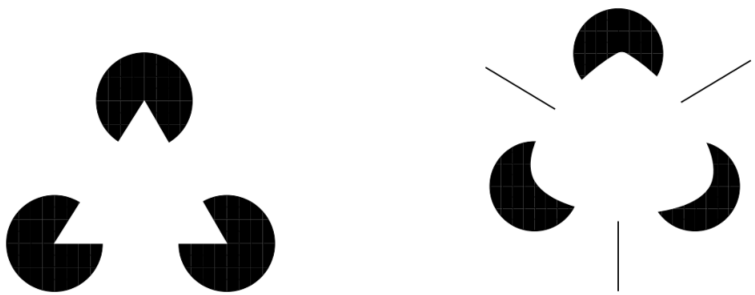


Wavelet zoom

Illusory contours

- It is rare that an image line has no hole in it. The brain compensate these defaults using more elaborate image analysis.

- Closing edge curves and understanding illusory contours requires computational models that are not as local as multiscale differential operators. Such contours can be obtained as the solution of a global optimization that incorporates constraints on the regularity of contours and takes into account the existence of occlusions.



The illusory edges of a straight and a curved triangle are perceived in domains where the images are uniformly white.

Wavelet Zoom

- Lipschitz Regularity
- Wavelet Transform Modulus Maxima
- Multiscale Edge Detection
- Multifractals

Wavelet zoom

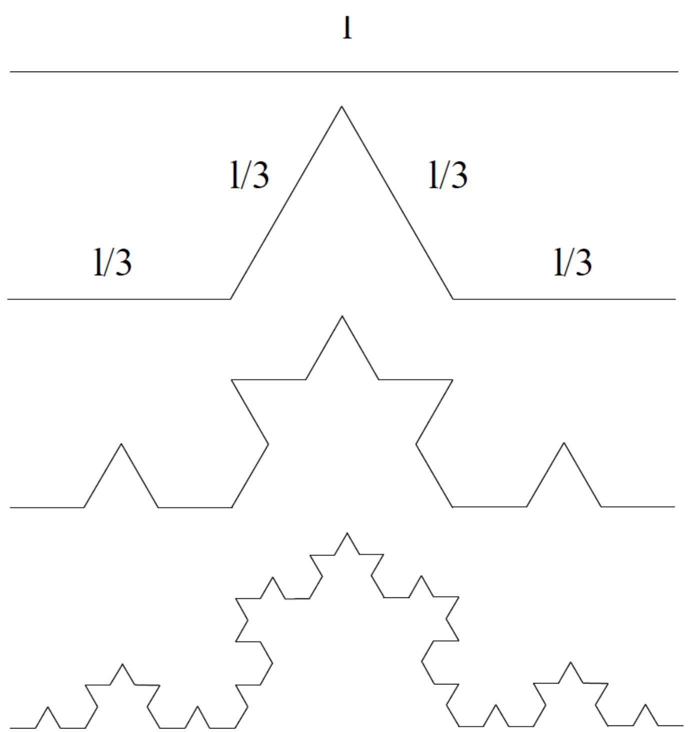
- Signals that are singular at almost every point were originally studied as pathological objects of pure mathematical interest. Such phenomena are encountered everywhere.

- Two important properties of multifractals are *self-similarity* and *non-integer dimension*.

- The singularities of multifractals often vary from point to point. Point-wise measurements of Lipschitz exponents are not possible because of the finite numerical resolution.

Fractal Sets and Self-Similar Functions

- A set S is self-similar if it is the union of disjoint subsets $S_1, \cdots, S_k$ that can be obtained from S with a scaling, translation and rotation. This self-similarity often implies an infinite multiplication of details, which creates irregular structures



The Von Koch curve:

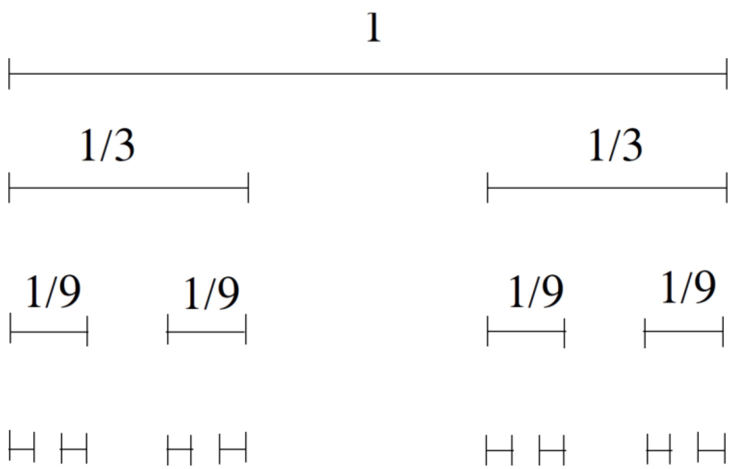
The fractal set is obtained by recursively dividing each segment of length l in four segments of length $l/3$.

Each division multiplies the length by 4/3, so the limit of these subdivisions is a curve of infinite length

The triadic Cantor set

The triadic Cantor set is constructed by recursively dividing intervals of size l in two subintervals of size $l/3$ and a central hole.

The iteration begins from $[0,1]$. The Cantor set obtained as a limit of these subdivisions is a dust of points in $[0,1]$.



Fractal Dimension

- The Von Koch curve has infinite length in a finite square of $\mathbb{R}^2$. The usual length measurement is not well adapted to characterize the topological properties of fractal curves.

- Let S be a bounded set in $\mathbb{R}^n$. We count the minimum number $N(s)$ of balls of radius s needed to cover S. If S is a set of dimension D with a finite length ($D = 1$), surface ($D = 2$), or volume ($D = 3$), then

$$N(s) \sim s^{-D}$$

$$D = -\lim_{s \to 0} \frac{\log N(s)}{\log s}$$

- The capacity dimension D of S generalizes this result is defined by

$$D = -\liminf_{s \to 0} \frac{\log N(s)}{\log s}$$

Fractal Dimension

- **Example 6.7:** The Von Koch curve has infinite length because its fractal dimension is $D > 1$. We need $N(s) = 4^n$ balls of size $s = 3^{-n}$ to cover the whole curve, thus,

$$N(3^{-n}) = (3^{-n})^{-\log 4/\log 3}$$

At any other scales, the minimum number of balls $N(s)$ to cover this curve satisfies

$$D = -\liminf_{s \to 0} \frac{\log N(s)}{\log s} = \frac{\log 2}{\log 3}$$

Wavelet zoom

Self-Similar Functions

- Let f be a continuous function with a compact support. f is self-similar if there exist disjoint subsets $S_1, \cdots, S_k$ such that the graph of f restricted to each S_i is an affine transformation of f.

  This means that there exist a scale $l_i > 1$, a translation r_i, a weight p_i, and a constant c_i such that

$$\forall t \in S_i, \qquad f(t) = c_i + p_i f\big(l_i(t - r_i)\big)$$

The wavelet transform and its modulus maxima of a self-similar function is also self-similar. Let g be an affine transformation of f

$$g(t) = pf\big(l(t - r)\big) + c$$

Wavelet transform: $Wg(u,s) = \int_{-\infty}^{+\infty} g(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt$

$$t' = l(t - r)$$

$$Wg(u,s) = \frac{p}{\sqrt{l}} Wf(l(u - r), sl)$$

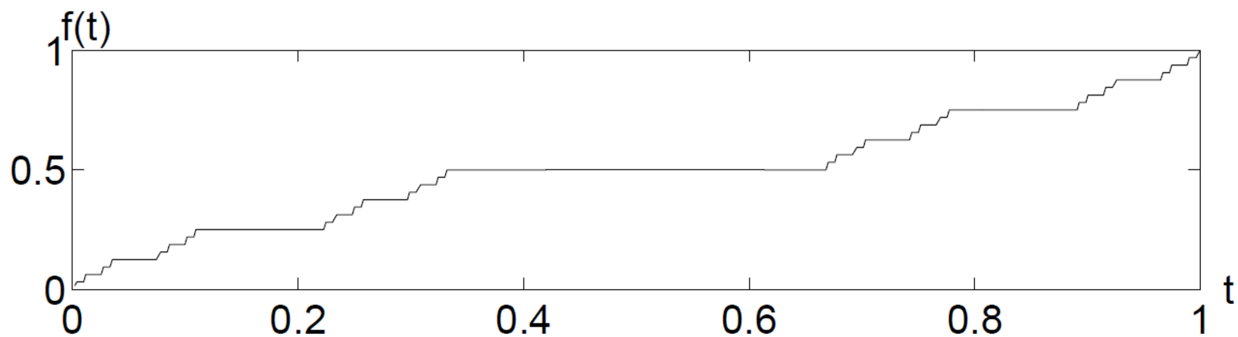
Self-Similar Functions

- **Example 6.10:** A devil's staircase is the integral of a Cantor measure:

$$f(t) = \int_0^t d\mu_\infty(x)$$

It is a continuous function that increases from 0 to 1 on $[0, 1]$. The recursive construction of the Cantor measure implies that f is self-similar

$$f(t) = \begin{cases} p_1 f(3t) & \text{if } t \in [0, 1/3] \\ p_1 & \text{if } t \in [1/3, 2/3] \\ p_1 + p_2 f(3t-2) & \text{if } t \in [2/3, 1] \end{cases}$$



Homework

Chapter 7: 7.12 and 7.14 (a) (b) (A Wavelet Tour of Signal Processing, 3rd edition)

Q & A

Many Thanks